

On effective mass of a photon in a strong magnetic field

V.M. Katkov

Budker Institute of Nuclear Physics,
Novosibirsk, 630090, Russia
e-mail: katkov@inp.nsk.su

March 18, 2014

Abstract

For the magnetic field in order of the Schwinger critical value or much larger it, the effective mass of a real photon with a preset polarization is investigated in the energy region including two lower creation thresholds of electron and positron on Landau levels. In the high-energy range, when the number of thresholds is large, the quasiclassical approach is used.

1. In 1971, Adler [1] had calculated the photon polarization operator in a constant and homogenous magnetic field using the proper-time technique developed by Schwinger [2]. The polarization operator on mass shell ($k^2 = 0$, the metric $ab = a^0b^0 - \mathbf{a}\mathbf{b}$ is used) in a strong magnetic field $H \gtrsim H_0 = m^2/e = 4,4 \cdot 10^{13}$ G (the system of units $\hbar = c = 1$ is used) was investigated well enough in the energy region lower the pair creation threshold (see, for example, the papers [3], [4] and the bibliography cited there). Here we consider the polarization operator for energies less than the third creation threshold of electron and positron on Landau levels. We investigate also the effective mass in the region of large threshold number using the quasiclassical approach. The general case of an arbitrary value of the photon energy and magnetic field strength, we shall consider in another work.

Our analysis is based on the general expression for the contribution of spinor particles to the polarization operator obtained in a diagonal form in [5] (see Eqs. (3.19), (3.33)). For the case of pure magnetic field we have in a covariant form the following expression

$$\Pi^{\mu\nu} = - \sum_{i=2,3} \kappa_i \beta_i^\mu \beta_i^\nu, \quad \beta_i \beta_j = - \delta_{ij}, \quad \beta_i k = 0; \quad (1)$$

$$\beta_2^\mu = (F^* k)^\mu / \sqrt{-(F^* k)^2}, \quad \beta_3^\mu = (F k)^\mu / \sqrt{-(F^* k)^2},$$

$$\text{Tr} F F^* = 0, \quad \text{Tr} F^2 = F^{\mu\nu} F_{\mu\nu} = 2(H^2 - E^2) \equiv 2f > 0, \quad (2)$$

where $F^{\mu\nu}$ – the electromagnetic field tensor, $F^{*\mu\nu}$ – dual tensor, k^μ – the photon momentum, $(Fk)^\mu = F^{\mu\nu}k_\nu$,

$$\kappa_i = \frac{\alpha}{\pi} m^2 r \int_{-1}^1 dv \int_0^{\infty - i0} f_i(v, x) \exp[i\psi(v, x)] dx. \quad (3)$$

Here

$$\begin{aligned} f_2(v, x) &= 2 \frac{\cos(vx) - \cos x}{\sin^3 x} - \frac{\cos(vx)}{\sin x} + v \frac{\cos x \sin(vx)}{\sin^2 x}, \\ f_3(v, x) &= \frac{\cos(vx)}{\sin x} - v \frac{\cos x \sin(vx)}{\sin^2 x} - (1 - v^2) \cot x, \\ \psi(v, x) &= \frac{1}{\mu} \left\{ 2r \frac{\cos x - \cos(vx)}{\sin x} + [r(1 - v^2) - 1]x \right\}; \\ r &= -(F^*k)^2 / 4m^2 f, \quad \mu^2 = f / H_0^2. \end{aligned} \quad (4)$$

The real part of κ_i determines the refractive index n_i of the photon with polarization $e_i = \beta_i$:

$$n_i = 1 - \frac{\text{Re} \kappa_i}{2\omega^2}. \quad (6)$$

At $r > 1$, the proper value of polarization operator κ_i includes the imaginary part which determines the probability per unit length of pair production by photon with the polarization β_i :

$$W_i = -\frac{1}{\omega} \text{Im} \kappa_i \quad (7)$$

For $r < 1$, the integration counter over x in Eq. (3) can be turn to the lower semiaxis ($x \rightarrow -ix$), then the value κ_i becomes real in an explicit form.

2. At $r < 1$, the expression for κ_i takes the following form:

$$\kappa_i = \alpha m^2 \frac{r}{\pi} \int_{-1}^1 dv \int_0^\infty F_i(v, x) \exp[-\chi(v, x)] dx, \quad (8)$$

Here

$$F_2(v, x) = \frac{1}{\sinh x} \left(2 \frac{\cosh x - \cosh(vx)}{\sinh^2 x} - \cosh(vx) + v \sinh(vx) \coth x \right), \quad (9)$$

$$F_3(v, x) = \frac{\cosh(vx)}{\sinh x} - v \frac{\cosh x \sinh(vx)}{\sinh^2 x} - (1 - v^2) \coth x; \quad (10)$$

$$\chi(v, x) = \frac{1}{\mu} \left[2r \frac{\cosh x - \cosh(vx)}{\sinh x} + (rv^2 - r + 1)x \right]. \quad (11)$$

For the energy sufficiently close to the threshold ($(1 - r)/\mu \ll 1$), we add to the integrand for κ_3 in (8) and take off the function

$$(1 - v^2) \exp[-\chi_{00}(v, x)], \quad \chi_{00}(v, x) = \frac{1}{\mu} [2r + (rv^2 - r + 1)x]. \quad (12)$$

Integrating over x the deducted part of the integrand, we have

$$\kappa_3^{00} = -\alpha m^2 \frac{\mu}{\pi} \exp\left(-\frac{2r}{\mu}\right) \int_{-1}^1 dv \frac{r(1 - v^2)}{rv^2 - r + 1}. \quad (13)$$

After integration over v , we recover κ_3 in the following well-behaved form:

$$\begin{aligned} \kappa_3 &= \kappa_3^1 + \kappa_3^{00}, \quad \kappa_3^1 = \alpha m^2 \frac{r}{\pi} \\ &\times \int_{-1}^1 dv \int_0^\infty \{F_3(v, x) \exp[-\chi(v, x)] + (1 - v^2) \exp[-\chi_{00}(v, x)]\} dx, \end{aligned} \quad (14)$$

$$\kappa_3^{00} = \alpha m^2 \frac{\mu}{\pi} \exp\left(-\frac{2r}{\mu}\right) [2 + B(r)]; \quad (15)$$

$$B(r) = \frac{2}{\sqrt{r(1-r)}} \arctan \sqrt{\frac{1-r}{r}} - \frac{\pi}{\sqrt{r(1-r)}}. \quad (16)$$

For superstrong fields ($\mu \gg 1$), the value $x \lesssim 1$ contributes in the integral for κ_2 and κ_3^1 and the exponential terms in the integrands can be substitute for unit. As a result, we have for the leading terms of expansion in series of μ :

$$\kappa_2 \simeq -\frac{4r}{3\pi} \alpha m^2, \quad \kappa_3 \simeq \alpha m^2 \frac{\mu}{\pi} (2 + B(r)). \quad (17)$$

Near the threshold, when $1 - r \ll 1$, $B(r) \simeq 2 - \pi/\sqrt{1-r}$, and we obtain:

$$\kappa_2 \simeq -\frac{4}{3\pi} \alpha m^2, \quad \kappa_3 \simeq -\alpha m^2 \mu \frac{1}{\sqrt{1-r}} \left(1 - \frac{4}{\pi} \sqrt{1-r}\right). \quad (18)$$

In low-energy range ($r \ll 1$), we have:

$$B(r) \simeq -2 - \frac{4}{3}r, \quad \kappa_2 \simeq -\frac{4r}{3\pi} \alpha m^2, \quad \kappa_3 \simeq -\frac{4r\mu}{3\pi} \alpha m^2. \quad (19)$$

Eq. (19) coincides with Eqs. (2.4), (2.9) in [3].

3. We go on to the next energy region, which upper boundary is higher the second threshold r_{10} (but not too close to the third threshold r_{20}). On this threshold, one of the particles is created on the first excited level and another – in the ground state. In general case

$$r_{lk} = (\varepsilon(l) + \varepsilon(k))^2 / 4m^2, \quad \varepsilon(l) = \sqrt{m^2 + 2eHl} = m\sqrt{1 + 2\mu l}. \quad (20)$$

For $1 < r < r_{10}$, the integration counter over x in Eq. (3) can be turn to the lower imaginary semiaxis, except the integrand term

$$- (1 - v^2) \cot x \exp[i\psi(v, x)]. \quad (21)$$

Let's add to Eq. (21) and take off the function

$$i(1 - v^2) \exp[i\psi_{\text{red}}(v, x)], \quad \psi_{\text{red}}(v, x) = \frac{1}{\mu} \{2ir + [r(1 - v^2) - 1]x\}. \quad (22)$$

$$(23)$$

For the sum of the functions, the integration counter over x can be turn to the lower semiaxis. For the residuary function, the integral over x has the following form

$$\begin{aligned} \int_0^\infty \exp[i\psi_{\text{red}}(v, x)] dx &= \exp\left(-\frac{2r}{\mu}\right) \frac{i\mu}{r(1 - v^2) - 1 + i0} \\ &= \mu \exp\left(-\frac{2r}{\mu}\right) \left[i \frac{\mathcal{P}}{r - 1 - rv^2} + \pi \delta(r - 1 - rv^2) \right]. \end{aligned} \quad (24)$$

The operator \mathcal{P} means the principal value integral. Carrying out the integration over v , we have after not complicated calculations

$$\begin{aligned} -ir \int_{-1}^1 dv (1 - v^2) \left[i \frac{\mathcal{P}}{r - 1 - rv^2} + \pi \delta(r - 1 - rv^2) \right] &= 2 + B(r), \\ B(r) &= \frac{2}{\sqrt{r(r - 1)}} \ln(\sqrt{r} + \sqrt{r - 1}) - \frac{i\pi}{\sqrt{r(r - 1)}}. \end{aligned} \quad (25)$$

Finally the expression for κ_3 takes the form of analytical extension of Eq. (14) into the region $r > 1$.

4. The integrals for κ_2 and κ_3^1 have the root divergence at $r = r_{10}$. To bring out these distinctions in an explicit form, let's consider the main asymptotic terms of corresponding integrand at $x \rightarrow \infty$:

$$\begin{aligned} \kappa_i^{10} &= \alpha m^2 r \frac{2}{\pi} \int_{-1}^1 dv \int_0^\infty d_i(v) \exp[-\chi_{10}(v, x)] dx, \\ d_2 &= v - 1, \quad d_3 = 1 - v - \frac{2r}{\mu}(1 - v^2) \end{aligned} \quad (26)$$

$$\chi_{10}(v, x) = \chi_{00}(v, x) + 1 - v = \frac{2r}{\mu} + \frac{1}{\mu} [(1 - v)\mu + rv^2 - r + 1] x. \quad (27)$$

After elementary integration over x , one gets

$$\kappa_i^{10} = \alpha m^2 \mu r \frac{2}{\pi} \exp\left(-\frac{2r}{\mu}\right) \int_{-1}^1 dv \frac{d_i(v)}{rv^2 - \mu v - r + 1 + \mu}. \quad (28)$$

Performing integration over v , we have:

$$\kappa_2^{10} = \alpha m^2 \mu r \frac{2}{\pi} \exp\left(-\frac{2r}{\mu}\right) \left[\frac{\mu/2r - 1}{\sqrt{h(r)}} A(r) - \frac{1}{2r} \ln(2\mu + 1) \right], \quad (29)$$

$$\begin{aligned} \kappa_3^{10} &= \alpha m^2 \mu r \frac{2}{\pi} \exp\left(-\frac{2r}{\mu}\right) \\ &\times \left[\frac{\mu/2r - 1 - 2/\mu}{\sqrt{h(r)}} A(r) - \frac{1}{2r} \ln(2\mu + 1) + \frac{2}{\mu} \right], \end{aligned} \quad (30)$$

$$\begin{aligned} A(r) &= \arctan \frac{r - \mu/2}{\sqrt{h(r)}} + \arctan \frac{r + \mu/2}{\sqrt{h(r)}} \\ &= \pi - \arctan \frac{\sqrt{h(r)}}{r - \mu/2} - \arctan \frac{\sqrt{h(r)}}{r + \mu/2}, \end{aligned} \quad (31)$$

$$h(r) = (1 + \mu)r - r^2 - \mu^2/4. \quad (32)$$

For $r = r_{10} = (1 + \mu + \sqrt{1 + 2\mu})/2$, $h(r) = 0$ and the values κ_i^{10} diverge at $r = r_{10}$:

$$\kappa_i^{10} \simeq -4\alpha m^2 r \exp\left(-\frac{2r}{\mu}\right) \frac{\beta_i}{\sqrt{h(r)}}, \quad \beta_2 = \frac{\mu}{2} - \frac{\mu^2}{4r}, \quad \beta_3 = 1 + \frac{\mu}{2} - \frac{\mu^2}{4r}. \quad (33)$$

For higher photon energies $r > r_{10}$, a new channel of pair creation arises, and Eq. (28) changes over (cf. (25)):

$$\begin{aligned} \kappa_i^{10} &= \alpha m^2 \mu r \frac{2}{\pi} \exp\left(-\frac{2r}{\mu}\right) \\ &\times \int_{-1}^1 dv d_i(v) \left[\frac{\mathcal{P}}{rv^2 - \mu v - r + 1 + \mu} - i\pi \delta(rv^2 - \mu v - r + 1 + \mu) \right]; \end{aligned} \quad (34)$$

At $r - r_{10} \ll 1$

$$\kappa_i^{10} \simeq -4i\alpha m^2 r \exp\left(-\frac{2r}{\mu}\right) \frac{\beta_i}{\sqrt{-h(r)}}. \quad (35)$$

This direct procedure of divergence elimination can be extended further.

5. For strong fields and high energy levels ($\mu \gtrsim 1$, $r \gg \mu$), the main contribution to the integral in Eq. (3) is given by small values of $x \sim (\mu/r)^{1/3} \ll 1$. Expanding the entering functions Eq. (4) over x , and carrying out the change of variable $x = \mu t$, we get:

$$\begin{aligned} \kappa_i &= \frac{\alpha m^2 \kappa^2}{24\pi} \int_0^1 \alpha_i(v) (1 - v^2) dv \int_0^\infty t \exp[-i(t + \xi \frac{t^3}{3})] dt; \quad \sqrt{\xi} = \frac{\kappa(1 - v^2)}{4}, \\ \alpha_2 &= 3 + v^2, \quad \alpha_3 = 2(3 - v^2), \quad \kappa^2 = 4r\mu^2 = -\frac{(Fk)^2}{m^2 H_0^2}. \end{aligned} \quad (36)$$

At $\kappa \gg 1$ ($\xi \gg 1$) the small t contributes to the integral (36) ($\xi t^3 \sim 1$), and in the argument of exponent Eq. (36) the linear over t term can be omit. The condition $\kappa \gg 1$ is identically valid in this case. Carrying out the change of variable:

$$\xi t^3/3 = -ix, \quad t = \exp\left(\frac{-i\pi}{6}\right) \left(\frac{3x}{\xi}\right)^{1/3}, \quad (37)$$

one obtains:

$$\kappa_i = \frac{\alpha m^2 \kappa^2}{24\pi} \exp\left(\frac{-i\pi}{3}\right) \frac{1}{3} \left(\frac{48}{\kappa^2}\right)^{2/3} \Gamma\left(\frac{2}{3}\right) \int_0^1 dv \alpha_i(v) (1-v^2)^{-1/3}. \quad (38)$$

After integration over v we have:

$$\begin{aligned} \kappa_i &= \frac{\alpha m^2 (3\kappa)^{2/3}}{7\pi} \frac{\Gamma^3\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} (1 - i\sqrt{3}) \beta_i \\ &= (0.175 - 0.304i) \beta_i \alpha m^2 \kappa^{2/3}, \quad \beta_2 = 1, \quad \beta_3 = 3/2. \end{aligned} \quad (39)$$

6. It follows from Eq. (39) that for $\alpha \kappa^{2/3} > 1$, the photon effective mass becomes larger than the mass of created electron and positron. And so, it seems that Eq. (39) is valid at the photon energy fulfilling the condition $\alpha \kappa^{2/3} \ll 1$. It should be noted that this expression does not depend on the electron mass. At the same time, the first order of the radiation correction to the electron mass have a form ($\chi = \varepsilon H_\perp / m H_0$, ε is the electron energy):

$$m_{\text{rad1}}^2 = 2D \alpha m^2 \chi^{2/3} = 2D \alpha \tilde{\chi}^{2/3}, \quad D = \frac{7(3)^{1/6}}{27} \Gamma\left(\frac{2}{3}\right) (1 - i\sqrt{3}) \quad (40)$$

$$= (0.422 - 0.730i), \quad \tilde{\chi}^2 = e^2 \mathcal{P} F^2 \mathcal{P}, \quad (41)$$

and does not depend on the mass too. The main term in the second order of the radiation correction to the mass has a form

$$m_{\text{rad2}}^2 = \frac{13\alpha^2 m^2 \chi}{36\sqrt{3}} \left[1 - i\frac{2}{\pi} \left(\ln \frac{\chi}{2\sqrt{3}} - C - \frac{142}{39} \right) \right] \quad (42)$$

$$= 0.2085\alpha^2 m^2 \chi [1 - 0.637i(\ln \chi - 5.461)], \quad (43)$$

where C – Euler's constant. This correction includes the additional factor $\sim \alpha \kappa^{1/3}$ comparing to Eq. (40), and formally can be larger the last. But for this value of parameter χ , one can not use the perturbation theory. In this case instead of the Dirac equation, we must use the well-known Schwinger equation

$$[\hat{\mathcal{P}} - m - M(\mathcal{P}, F)]\psi = 0, \quad \hat{\mathcal{P}} = \gamma^\mu \mathcal{P}_\mu \equiv \gamma \mathcal{P}, \quad \mathcal{P}_\mu = i\frac{\partial}{\partial x^\mu} - eA_\mu, \quad (44)$$

where M is the mass operator including, generally speaking, the all series of the perturbation theory. At substitution m_{rad1} in place of m into $\alpha\kappa^{1/3}$, we have a value $\sim \sqrt{\alpha}/5$, not depending on any parameter. This value is small and therefore, we can expect that the first order of the perturbation theory contributes mainly into the mass operator M . In this order, the mass operator have a relatively simple form

$$M_1(\mathcal{P}, F) \simeq D\alpha\tilde{\chi}^{-4/3}(e^2\gamma F^2\mathcal{P}). \quad (45)$$

Multiply Eq. (44) by the operator $\hat{\mathcal{P}} + m - M$ and take into account that

$$\{\hat{\mathcal{P}}, e^2\gamma F^2\mathcal{P}\} = 2e^2\mathcal{P}F^2\mathcal{P} = 2\tilde{\chi}^2,$$

and the term $\propto e^2\gamma F^2\mathcal{P}$ can be omit (that relative value $\sim m_{\text{rad1}}^2/\varepsilon^2$). As a result we have the following squared equation

$$(\hat{\mathcal{P}}^2 - m^2 - 2D\alpha\tilde{\chi}^{2/3})\psi = (\hat{\mathcal{P}}^2 - m^2 - m_{\text{rad1}}^2)\psi = 0 \quad (46)$$

All stated above is valid under condition $\varepsilon e H_{\perp} \gg (m^2 + m_{\text{rad1}}^2)^{3/2}$. For $m_{\text{rad1}}^2 > m^2$, this condition changes over $1 \gg \alpha^{3/2}$.

The work was supported by the Ministry of Education and Science of the Russian Federation.

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